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A THEOREM ON ALMOST PERIODIC FUNCTIONS OF INFINITELY MANY VARIABLES

BY

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1. Introduction.

For an almost periodic function $f(x)$ of one variable the following theorem is true: *the function is periodic if and only if the set of all translated functions $f(x+h)$ is closed with respect to uniform convergence.* As remarked by B. JESSEN this can easily be proved directly from the structure definition of almost periodicity; also for the corresponding theorem on almost periodic functions $f(x_1, x_2, \cdots, x_m)$ in the m-dimensional space he has shown me such a proof.

The way in which one generalizes the word "periodic" when passing from 1 to m dimensions with this theorem is to what might be called "fully periodic". Let $f(x_1, x_2, \cdots, x_m)$ be a continuous function of $(x_1, x_2, \cdots, x_m)$. A vector $(h_1, h_2, \cdots, h_m)$ is called a period vector of $f(x_1, x_2, \cdots, x_m)$ if $f(x_1 + h_1, x_2 + h_2, \cdots, x_m + h_m) = f(x_1, x_2, \cdots, x_m)$ for all $(x_1, x_2, \cdots, x_m)$. The set of all period vectors of $f(x_1, x_2, \cdots, x_m)$ is obviously a closed module (a module being a set which with two points also contains their sum and difference). This module may consist of $(0, 0, \cdots, 0)$ only, and the function is not periodic at all. If the dimension of the module is equal to the dimension m of the space we call the function *fully periodic*. Our theorem for almost periodic functions of m variables can then be stated, *such a function is fully periodic if and only if the set of all translated functions $f(x_1 + h_1, x_2 + h_2, \cdots, x_m + h_m)$ is closed with respect to uniform convergence.*

JESSEN put the problem to decide whether this theorem also holds for almost periodic functions of an infinite number of variables. It will turn out—as a result of this paper—that it does hold in verbally the same formulation, the word "fully periodic" needing of course an appropriate definition. Incidentally, we shall get another proof in the m-dimensional case than the one referred to above.

JESSEN also remarked that if the almost periodic function $f(x_1, x_2, \cdots)$ in question is limit-periodic, the theorem is true as a simple consequence of a result of H. BOHR [2], [3] obtained in connection with a study of certain classes of almost periodic functions (see **4**, and **6**, p. 12 of the present paper). BOHR's result concerned an infinite system of linear congruences

$$r_{n1} x_1 + r_{n2} x_2 + \cdots + r_{nq_n} x_{q_n} \equiv \theta_n \pmod 1, n = 1, 2, \cdots$$

with infinitely many real variables $x_1, x_2, \cdots$ and rational coefficients. It was later on generalized by BOHR and FØLNER [4] to arbitrary coefficients, and this generalization will be a tool for the proof of the general case of our theorem.

2. Almost Periodic Functions of Infinitely Many Variables.

We start with recalling that an almost periodic function $f(x_1, x_2, \cdots)$ of an infinite number of real variables $x_1, x_2, \cdots$[1] can be characterized as a (complex-valued) function, defined on the space $\Re^\infty$ of points $\boldsymbol{x} = (x_1, x_2, \cdots)$, which can be uniformly approximated by trigonometric polynomials

$$\sum c_n e^{2\pi i (a_1^n x_1 + a_2^n x_2 + \cdots + a_{q_n}^n x_{q_n})}.$$

The space $\Re^\infty$ is topologized in the following way. A sequence of points $\boldsymbol{x}^n$ is said to converge towards $\boldsymbol{x}$, if $x_1^n \to x_1, x_2^n \to x_2, \cdots$. Thus every trigonometric polynomial, and hence also every almost periodic function is continuous on $\Re^\infty$.

The exponent vectors $(a_1^n, a_2^n, \cdots, a_{q_n}^n, 0, 0, \cdots)$ in the trigonometric polynomial above have zeros on all coordinate places from a certain number. We define the space $\Re_\infty$ as the set of all vectors $\boldsymbol{a} = (a_1, a_2, \cdots)$ with zeros on all coordinate places from a certain number (depending on the point), and so the exponent vectors can be said to belong to $\Re_\infty$. The inner

[1] S. BOCHNER [1].

product $a_1 x_1 + a_2 x_2 + \cdots$ between a vector $\boldsymbol{a} = (a_1, a_2, \cdots)$ from $\mathfrak{R}_\infty$ and a vector $\boldsymbol{x} = (x_1, x_2, \cdots)$ from $\mathfrak{R}^\infty$ is denoted by $\boldsymbol{a} \cdot \boldsymbol{x}$. The trigonometric polynomial above may then be written

$$\sum{}' c_n e^{2\pi i \boldsymbol{a}^n \cdot \boldsymbol{x}} \qquad (\boldsymbol{a}^n \varepsilon \mathfrak{R}_\infty, \boldsymbol{x} \varepsilon \mathfrak{R}^\infty).$$

To every almost periodic function $f(\boldsymbol{x})$ is associated a unique Fourier series

$$f(\boldsymbol{x}) \sim \sum{}' A_n e^{2\pi i \boldsymbol{a}^n \cdot \boldsymbol{x}}.$$

To the translated function $f(\boldsymbol{x} + \boldsymbol{h})$ is associated the Fourier series

$$f(\boldsymbol{x} + \boldsymbol{h}) \sim \sum{}' A_n e^{2\pi i \boldsymbol{a}^n \cdot \boldsymbol{h}} e^{2\pi i \boldsymbol{a}^n \cdot \boldsymbol{x}}.$$

A necessary and sufficient condition for a sequence of translated functions $f(\boldsymbol{x} + \boldsymbol{h}^1), f(\boldsymbol{x} + \boldsymbol{h}^2), \cdots$ to converge uniformly to the (*eo ipso*) almost periodic function $g(\boldsymbol{x})$ is that the Fourier series of $f(\boldsymbol{x} + \boldsymbol{h}^n)$ converge formally to the Fourier series of $g(\boldsymbol{x})$. Therefore the necessary and sufficient condition for the set of all translated functions $f(\boldsymbol{x} + \boldsymbol{h})$ to be closed is that the set of points

$$\left\{ \begin{array}{c} e^{2\pi i \boldsymbol{a}^1 \cdot \boldsymbol{h}} \\ e^{2\pi i \boldsymbol{a}^2 \cdot \boldsymbol{h}} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\}$$

where $\boldsymbol{h}$ runs through all vectors in $\mathfrak{R}^\infty$, be closed in $\mathfrak{R}^\infty$ or—what comes to the same thing—that the module of points

$$\left\{ \begin{array}{c} \boldsymbol{a}^1 \cdot \boldsymbol{h} \\ \boldsymbol{a}^2 \cdot \boldsymbol{h} \\ \cdot \\ \cdot \\ \cdot \end{array} \right\} + i_1 \left\{ \begin{array}{c} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \end{array} \right\} + i_2 \left\{ \begin{array}{c} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \end{array} \right\} + \cdots$$

where $\boldsymbol{h}$ runs through all vectors in $\mathfrak{R}^\infty$ and $i_1, i_2, \cdots$ through all integers, be closed in $\mathfrak{R}^\infty$.

3. Closed Modules and Substitutions in $\mathfrak{R}^\infty$.

Let $f(x_1, x_2, \cdots) = f(\boldsymbol{x})$ be a continuous function on $\mathfrak{R}^\infty$. A vector $\boldsymbol{h} = (h_1, h_2, \cdots)$ is called a period vector of $f(\boldsymbol{x})$ if $f(\boldsymbol{x}+\boldsymbol{h}) = f(\boldsymbol{x})$ for all points $\boldsymbol{x}$. The set of all period vectors of $f(\boldsymbol{x})$ is obviously a closed module in $\mathfrak{R}^\infty$. If it is not contained in any proper vector-subspace (i. e. *closed* linear subset[1]) of $\mathfrak{R}_\infty$ the function $f(\boldsymbol{x})$ is called fully periodic.

As a general example of a fully periodic function we consider an almost periodic function

$$f(\boldsymbol{x}) \sim \sum A_n e^{2\pi i \boldsymbol{a}^n \cdot \boldsymbol{x}}$$

which is periodic with the period 1 in all the coordinates $x_1, x_2, \cdots$ or—what comes to the same thing—a function which has all integral points as period vectors. Expressed as a property of the Fourier series this means that all the exponent vectors $\boldsymbol{a}^n$ are integral vectors.

In the previously cited paper [4]—and in more detail in [5]—a structure theorem for closed modules in $\mathfrak{R}^\infty$ was obtained. Incidentally, let me remark that this was done by introducing a suitable convergence notion in $\mathfrak{R}_\infty$ to the effect that a duality took place between the closed modules in $\mathfrak{R}^\infty$ and the closed modules in $\mathfrak{R}_\infty$. To formulate the structure theorem we have to introduce the notion of "substitution" in $\mathfrak{R}^\infty$. By this is understood a linear transformation $\boldsymbol{x} = T\boldsymbol{y}$ of the form

$$\begin{aligned} x_1 &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1p_1}y_{p_1} \\ x_2 &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2p_2}y_{p_2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \end{aligned}$$

which establishes a one-to-one mapping $\boldsymbol{x} \leftrightarrow \boldsymbol{y}$ of the whole infinite-dimensional space on the whole infinite-dimensional space. It turns out to be the same as a linear, bicontinuous, one-to-one mapping of $\mathfrak{R}^\infty$ onto itself. The structure theorem now runs as follows.

[1] A linear set in $\mathfrak{R}^\infty$ need not be closed as is the case in the m-dimensional space. For instance the points in $\mathfrak{R}_\infty$ form a linear subset of $\mathfrak{R}^\infty$ and its closure is the whole $\mathfrak{R}^\infty$.

Structure theorem. *A closed module in the infinite-dimensional space* $\mathfrak{R}^\infty$ *is a point set E which by a substitution can be transformed into a point set of a special form, namely a point set* $[(x_1, x_2, \cdots)]$ *of the following structure: The indices* $1, 2, \cdots, n, \cdots$ *can be divided into three fixed classes* $\{n_r\}, \{n_s\}, \{n_t\}$, *such that the coordinates* x_{n_r} *independently run through all numbers, and the coordinates* x_{n_s} *independently run through all integers, while all the remaining coordinates* x_{n_t} *are constantly zero. Conversely, each such point set E is a closed module.*

The vector-subspaces (i. e. closed linear subsets) of $\mathfrak{R}^\infty$ are of course characterized by an empty class $\{n_s\}$.

That a closed module is not contained in any proper vector-subspace of $\mathfrak{R}^\infty$—as is the demand to the period module of a fully periodic function—means that it has an empty class $\{n_t\}$ or, what is equivalent, that there exists a transformed set which contains all integral points.

If $f(\boldsymbol{x})$ is an almost periodic function on $\mathfrak{R}^\infty$ and we subject $\boldsymbol{x}$ to the substitution $\boldsymbol{x} = T\boldsymbol{y}$, then the transformed function $f(T\boldsymbol{y})$ is an almost periodic function of $\boldsymbol{y}$ in $\mathfrak{R}^\infty$. In fact, a trigonometric polynomial in $\boldsymbol{x}$ will be transformed into a trigonometric polynomial in $\boldsymbol{y}$ by this process, and, to complete the reasoning, $|f(\boldsymbol{x}) - s(\boldsymbol{x})| \leq \varepsilon$ for all $\boldsymbol{x}$ means the same as $|f(T\boldsymbol{y}) - s(T\boldsymbol{y})| \leq \varepsilon$ for all $\boldsymbol{y}$. Furthermore, if $f(\boldsymbol{x})$ has the Fourier series

$$\sum A_n e^{2\pi i \boldsymbol{a}^n \cdot \boldsymbol{x}}, \tag{1}$$

then $f(T\boldsymbol{y})$ has the formally transformed Fourier series

$$\sum A_n e^{2\pi i \boldsymbol{b}^n \cdot \boldsymbol{y}}, \tag{2}$$

where $\boldsymbol{b}^n$ is determined by $\boldsymbol{a}^n \cdot T\boldsymbol{y} = \boldsymbol{b}^n \cdot \boldsymbol{y}$.[1] To see this, let $s_n(\boldsymbol{x})$ be a sequence of trigonometric polynomials which converges uniformly to $f(\boldsymbol{x})$. It will converge formally to the Fourier series (1), and therefore the sequence $s_n(T\boldsymbol{y})$ of trigonometric

[1] Incidentally we remark that the $\boldsymbol{b}^n$'s are obtained from the $\boldsymbol{a}^n$'s by a "substitution" T^* in $\mathfrak{R}_\infty$, T^* being the "adjoint substitution" of T (see [5]).

polynomials in $\boldsymbol{y}$ converging uniformly to $f(T\boldsymbol{y})$ will converge formally to the series (2), which must therefore be the Fourier series of $f(T\boldsymbol{y})$.

Obviously, a period vector $\boldsymbol{h}$ of $f(\boldsymbol{x})$ is transformed into a period vector $\boldsymbol{k}$ ($\boldsymbol{h} = T\boldsymbol{k}$) of the function $f(T\boldsymbol{y})$.

It follows from what has been said that a necessary and sufficient condition for an almost periodic function $f(\boldsymbol{x})$ to be fully periodic is that there exists a substitution $\boldsymbol{x} = T\boldsymbol{y}$ such that $f(T\boldsymbol{y})$ is periodic with the period 1 in all the coordinates $y_1, y_2, \cdots$, or—what comes to the same thing—that all the Fourier exponent vectors of $f(T\boldsymbol{y})$ are integral.

4. Infinitely Many Linear Congruences with Infinitely Many Variables.

We consider an arbitrary enumerable system of linear congruences with an enumerable number of real variables

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n_1}x_{n_1} \equiv \theta_1 \pmod 1\\
(3)\qquad &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n_2}x_{n_2} \equiv \theta_2 \pmod 1\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

or, briefly written,

$$
\begin{aligned}
&\boldsymbol{a}^1\cdot\boldsymbol{x} \equiv \theta_1\\
(3)\qquad &\boldsymbol{a}^2\cdot\boldsymbol{x} \equiv \theta_2\\
&\cdots\cdots\cdots\cdots,
\end{aligned}
$$

where every congruence only contains a finite number of variables and the a's and the θ's are arbitrary (real) numbers. By π_1 we denote the module of points $(\theta_1, \theta_2, \cdots)$ in $\mathfrak{R}^\infty$ for which the corresponding infinite system (3) has a solution, and by π_2 the module of points $(\theta_1, \theta_2, \cdots)$ for which any finite subsystem of (3) has a solution. The result of Bohr referred to in **1** was that in case of rational a's *a necessary and sufficient condition on the linear forms* $\boldsymbol{a}^n\cdot\boldsymbol{x}$ *in* (3) *in order that* $\pi_1 = \pi_2$ *is that there exists a substitution* $\boldsymbol{x} = T\boldsymbol{y}$ *in* $\mathfrak{R}^\infty$ *which transforms them into linear forms with integral coefficients.* The generalization in [4] of this result is that without the said restriction of rationality on the a's *a necessary and sufficient condition on the linear forms in* (3) *in order that* $\pi_1 = \pi_2$ *is that there exists a substitution in* $\mathfrak{R}^\infty$ *which transforms them into a system of linear forms of a certain simple type, denoted by* S. By a system of linear forms *of the type* S we understand a system where certain of the variables (finite or infinite in number) have mere integral coefficients while each of the remaining variables (finite or infinite in number) necessarily becomes 0 if for a sufficiently large m one solves the m first "zero-congruences" corresponding to the linear forms, i. e. the congruences (3) with $\theta_1 = \theta_2 = \cdots = 0$.

By the proof of this theorem the structure theorem for closed modules in $\mathfrak{R}^\infty$ played an important role.

5. Modules in the m-dimensional Space.

The duality between $\mathfrak{R}^\infty$ and $\mathfrak{R}_\infty$ loosely referred to above has its simpler origin in a duality considered by M. Riesz [6] between two m-dimensional spaces $R_m = \{(x_1, x_2, \cdots, x_m)\}$ and $R_m = \{(a_1, a_2, \cdots, a_m)\}$. Since we shall use it in our proof we shall state this duality explicitly[1]. To an arbitrary module M in R_m Riesz considers the point set in (the other space) R_m consisting of all points $\boldsymbol{a} = (a_1, a_2, \cdots, a_m)$ from this latter R_m for which

$$\boldsymbol{a} \cdot \boldsymbol{x} = a_1 x_1 + a_2 x_2 + \cdots + a_m x_m \equiv 0 \pmod 1$$

for every point $\boldsymbol{x} = (x_1, x_2, \cdots, x_m)$ from M. This point set is a closed module in R_m and is called the dual module of M. We denote it by M'. If we repeat the operation of passing to the dual module we get a closed module $M'' = (M')'$ in (the original space) R_m. The relation between M and M'' appears from the following important theorem.

Riesz's theorem. *If M is an arbitrary module in R_m, the dual M'' of its dual module M' is the closure $\overline{M}$ of M, i. e.*

$$M'' = \overline{M}.$$

Before passing to the real proof of our theorem on almost periodic functions of infinitely many variables we shall make a

[1] For the proofs, see [5].

simple, and rather obvious, remark about the type of module M which is obtained by placing subspaces of the same dimension $\nu < m$ through all integral points in R_m and parallel to the same vector-subspace R of dimension ν, i. e. the module obtained by adding to the points in R all integral points in R_m. Our statement is that the largest vector-space contained in M[1] is R. This follows, by an indirect reasoning, from the fact that a usual space of dimension > 0 contains more than an enumerable number of points.

6. Proof of the Theorem.

We repeat the theorem to be proved.

Theorem. *An almost periodic function* $f(\boldsymbol{x}) = f(x_1, x_2, \cdots)$ *of an infinite number of variables is fully periodic*[2] *if and only if the set of translated functions* $f(\boldsymbol{x}+\boldsymbol{h}) = f(x_1+h_1, x_2+h_2, \cdots)$ *is closed with respect to uniform convergence.*

It is plain that this theorem contains the analogous theorem on almost periodic functions of a finite number of variables as a special case.

1. The one part of the theorem is easily dealt with. Let $f(\boldsymbol{x})$ be a fully periodic function. In order to show that the set of translated functions $f(\boldsymbol{x}+\boldsymbol{h})$ is closed we choose a substitution $\boldsymbol{x} = T\boldsymbol{y}$ such that the transformed function $f(T\boldsymbol{y}) = g(\boldsymbol{y}) = g(y_1, y_2, \cdots)$ is periodic with the period 1 in all the coordinates $y_1, y_2, \cdots$ (see end of **3**). Let $\boldsymbol{h}^1, \boldsymbol{h}^2, \cdots$ be a sequence of vectors such that $f(\boldsymbol{x}+\boldsymbol{h}^n)$ converges uniformly to a function $d(\boldsymbol{x})$. We have to show that $d(\boldsymbol{x})$ is equal to a translated function $f(\boldsymbol{x}+\boldsymbol{h})$. Let $\boldsymbol{k}^n$ be the corresponding points of $\boldsymbol{h}^n$ by the substitution above, i. e. $\boldsymbol{h}^n = T\boldsymbol{k}^n$. Then $g(\boldsymbol{y}+\boldsymbol{k}^n)$ will converge uniformly to $d(T\boldsymbol{y}) = j(\boldsymbol{y})$. We only have to prove that $j(\boldsymbol{y}) = g(\boldsymbol{y}+\boldsymbol{k})$ for some $\boldsymbol{k}$ since then $d(\boldsymbol{x}) = j(T^{-1}\boldsymbol{x}) = g(T^{-1}\boldsymbol{x}+\boldsymbol{k}) = g(T^{-1}\boldsymbol{x}+T^{-1}\boldsymbol{h}) = f(\boldsymbol{x}+\boldsymbol{h})$, where $\boldsymbol{h}$ is determined by $\boldsymbol{h} = T\boldsymbol{k}$.

The function $g(\boldsymbol{y}) = g(y_1, y_2, \cdots)$ is periodic with the period 1 in all the coordinates, so we may assume that our

[1] The vector-space generated by all vector-spaces contained in M (which on account of the module property must be contained in M).

[2] See **3**.

$\boldsymbol{k}^n$-points are all lying in the "periodicity parallelotope" $0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1, \cdots$. This point set, however, is compact in the sense that every sequence $\boldsymbol{k}^1, \boldsymbol{k}^2, \cdots$ of points from this set has a subsequence which converges to a point of the set. As our $\boldsymbol{k}$ we may take any such limit-point of the sequence $\boldsymbol{k}^1, \boldsymbol{k}^2, \cdots$, for on account of the continuity of $g(\boldsymbol{y}), j(\boldsymbol{y}) = \lim g(\boldsymbol{y} + \boldsymbol{k}^n) = g(\boldsymbol{y} + \boldsymbol{k})$, q. e. d.

2. We now pass to the more difficult part of the theorem. Let $f(\boldsymbol{x})$ be an almost periodic function for which the set of translated functions $f(\boldsymbol{x} + \boldsymbol{h})$ is closed. Our task is to prove that $f(\boldsymbol{x})$ is fully periodic or, what comes to the same thing, that there exists a substitution $\boldsymbol{x} = T\boldsymbol{y}$ such that the transformed function $f(T\boldsymbol{y})$ has mere integral Fourier exponent vectors (see end of **3**). If

$$f(\boldsymbol{x}) \sim \sum A_n e^{2\pi i \boldsymbol{a}^n \cdot \boldsymbol{x}}, \text{ then } f(T\boldsymbol{y}) \sim \sum A_n e^{2\pi i \boldsymbol{b}^n \cdot \boldsymbol{y}},$$

where the linear forms $\boldsymbol{b}^n \cdot \boldsymbol{y}$ are obtained from the linear forms $\boldsymbol{a}^n \cdot \boldsymbol{x}$ by the substitution $\boldsymbol{x} = T\boldsymbol{y}$. So we have to find a substitution T which transforms the linear forms $\boldsymbol{a}^n \cdot \boldsymbol{x}$ into linear forms with integral coefficients. For this purpose we consider the corresponding system of congruences

$$\begin{aligned} \boldsymbol{a}^1 \cdot \boldsymbol{x} &\equiv \theta_1 \\ \boldsymbol{a}^2 \cdot \boldsymbol{x} &\equiv \theta_2 \\ &\cdots\cdots\cdots\cdots \end{aligned} \tag{4}$$

Let π_1 and π_2 be the modules defined in **4** corresponding to these congruences. Our assumption that the set of translated functions $f(\boldsymbol{x} + \boldsymbol{h})$ is closed is equivalent to π_1 being closed (see **2**).

It is plain that $\pi_1 \subseteq \pi_2$. Furthermore, the closure $Cl(\pi_1)$ of π_1 contains π_2, $Cl(\pi_1) \supseteq \pi_2$. For the proof of this let $(\theta_1^0, \theta_2^0, \cdots)$ be an arbitrary point from π_2. In order to approximate $(\theta_1^0, \theta_2^0, \cdots)$ by a point from π_1 we solve the m first congruences in (4) for this choice of $(\theta_1, \theta_2, \cdots)$ and a "large" m. Let $\boldsymbol{x}^0$ be a solution. Then $(\theta_1^0, \theta_2^0, \cdots, \theta_m^0, \boldsymbol{a}^{m+1} \cdot \boldsymbol{x}^0, \boldsymbol{a}^{m+2} \cdot \boldsymbol{x}^0, \cdots)$ from π_1 will be a "good" approximation to $(\theta_1^0, \theta_2^0, \cdots)$ in $\mathfrak{R}^\infty$. Since π_1 closed means $Cl(\pi_1) = \pi_1$, the relations

$$\pi_1 \subseteq \pi_2 \subseteq Cl(\pi_1)$$

imply $\pi_1 = \pi_2$. In case $f(\boldsymbol{x})$ is limit-periodic, i. e. all the $\boldsymbol{x}$-vectors are rational, the theorem of BOHR stated in **4** therefore immediately completes the proof.

In the general case we conclude by the theorem of BOHR and FØLNER in **4** that there exists a substitution $\boldsymbol{x} = T\boldsymbol{y}$ which transforms the linear forms into a system of the type S. Our proof will be completed if we can show that the new system has mere integral coefficients. Let the new system of congruences be

$$\begin{aligned} \boldsymbol{b}^1 \cdot \boldsymbol{y} &\equiv \theta_1 \\ \boldsymbol{b}^2 \cdot \boldsymbol{y} &\equiv \theta_2 \\ \cdots&\cdots\cdots . \end{aligned} \tag{5}$$

For an arbitrary m we consider the module of points $(\theta_1, \theta_2, \cdots, \theta_m)$ which is supplied by the m first congruences when $\boldsymbol{y}$ runs through all points in $\mathfrak{R}^\infty$, and we wish to show that this module is closed in R_m. We assume to the contrary that it is not closed. Then there exists a point $(\theta_1', \theta_2', \cdots, \theta_m')$ which belongs to its closure but not to the module itself. We choose a sequence $\boldsymbol{y}^r$ such that $(\boldsymbol{b}^1 \cdot \boldsymbol{y}^r, \boldsymbol{b}^2 \cdot \boldsymbol{y}^r, \cdots, \boldsymbol{b}^m \cdot \boldsymbol{y}^r) \to (\theta_1', \theta_2', \cdots \theta_m') \pmod 1$ and next by the diagonal procedure a subsequence of $\boldsymbol{y}^r$ such that $\boldsymbol{b}^{m+1} \cdot \boldsymbol{y}^r$ modulo 1 converges to a θ_{m+1}', $\boldsymbol{b}^{m+2} \cdot \boldsymbol{y}^r$ modulo 1 converges to a θ_{m+2}', etc., ad inf. The point $(\theta_1', \theta_2', \cdots)$ will then obviously belong to $Cl(\pi_1)$ but not to π_1 (since not even the m first congruences can be fulfilled for this $(\theta_1', \theta_2', \cdots)$). This is a contradiction since $Cl(\pi_1) = \pi_1$. Hence the module $\{(\theta_1, \theta_2, \cdots, \theta_m)\}$ is closed. It will therefore be sufficient to prove that if the system of linear forms in (5) has not mere integral coefficients, then the module cannot be closed for all m.

To fix the matter assume for instance that the first column in our system of linear forms be not integral. The property of a system of linear forms of the type S (see **4**) then implies that the variable y_1 necessarily becomes 0 if for a sufficiently large m one solves the m first "zero-congruences" corresponding to the linear forms, i. e. the congruences (5) with $\theta_1 = \theta_2 = \cdots = 0$. Let m be chosen in this way. We shall then show that the module $M = \{(\theta_1, \theta_2, \cdots, \theta_m)\}$ supplied by the m first congruences is not closed. To do this we write the m first congruences in (5) more fully as

$$\begin{aligned} b_{11}y_1 + b_{12}y_2 + \cdots + b_{1n}y_n &\equiv \theta_1 \\ b_{21}y_1 + b_{22}y_2 + \cdots + b_{2n}y_n &\equiv \theta_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots \\ b_{m1}y_1 + b_{m2}y_2 + \cdots + b_{mn}y_n &\equiv \theta_m. \end{aligned}$$

The column vectors in this system span a vector-space R of points

$$y_1\begin{Bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{m1} \end{Bmatrix} + y_2\begin{Bmatrix} b_{12} \\ b_{22} \\ \cdot \\ \cdot \\ \cdot \\ b_{m2} \end{Bmatrix} + \cdots + y_n\begin{Bmatrix} b_{1n} \\ b_{2n} \\ \cdot \\ \cdot \\ \cdot \\ b_{mn} \end{Bmatrix},$$

where $y_1, y_2, \cdots, y_n$ run through all numbers. Let its dimension be ν. The property that y_1 necessarily becomes zero by solution of the zero-congruences has then as geometrical consequence that the dimension of the lattice of integral points in R is $\leq \nu - 1$ (i. e. smaller than the dimension of R), for by the first property all integral points in R have to lie in the vector-subspace

$$y_2\begin{Bmatrix} b_{12} \\ b_{22} \\ \cdot \\ \cdot \\ \cdot \\ b_{m2} \end{Bmatrix} + \cdots + y_n\begin{Bmatrix} b_{1n} \\ b_{2n} \\ \cdot \\ \cdot \\ \cdot \\ b_{mn} \end{Bmatrix}$$

and the point

$$\begin{Bmatrix} b_{11} \\ b_{21} \\ \cdot \\ \cdot \\ \cdot \\ b_{m1} \end{Bmatrix}$$

has to lie outside this subspace[1]. The module M is obtained by adding to the points in R all integral points in R_m. Denoting the module of integral points in R_m

[1] In fact, if the point lay in the subspace we could even solve the zero-*equations* with an arbitrarily chosen y_1.

$$\begin{Bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \\ i_m \end{Bmatrix} = i_1 \begin{Bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} + i_2 \begin{Bmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{Bmatrix} + \cdots + i_m \begin{Bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{Bmatrix}$$

by I we may write this

$$M = R + I.$$

In **5** we have seen that the largest vector-subspace in M is R. We now show that the closure $\overline{M}$ of M contains a larger vector-subspace, i. e. a space of dimension $\geqq \nu + 1$. Then $\overline{M}$ must be different from M, and M cannot be closed as we had to prove.

We pass to the dual module M' of M (see **5**) which obviously consists of all integral points $(k_1, k_2, \cdots, k_m)$ in R_m that are orthogonal to R (for $(k_1, k_2, \cdots, k_m) \cdot (\underset{1}{0}, \underset{2}{0}, \cdots, 0, \underset{r}{1}, 0, \cdots \underset{m}{0}) \equiv 0$ means k_r integral, and $(k_1, k_2, \cdots, k_m) \cdot y_r (b_{1r}, b_{2r}, \cdots, b_{mr}) \equiv 0$ for all y_r means $(k_1, k_2, \cdots, k_m) \cdot (b_{1r}, b_{2r}, \cdots, b_{mr}) = 0$).

The vector-space R^* of arbitrary points in R_m which are orthogonal to R has the dimension $m - \nu$. Suppose that the module $M' \subseteqq R^*$ had the same dimension; then R could be characterized as the orthogonal vector-space to the module M'. But M' consists of integral points, so in particular a generating system $\boldsymbol{l}^1, \boldsymbol{l}^2, \cdots, \boldsymbol{l}^{m-\nu}$ is integral. Hence R would be the set of points $\boldsymbol{x}$ satisfying

$$\begin{aligned} \boldsymbol{l}^1 \cdot \boldsymbol{x} &= 0 \\ \boldsymbol{l}^2 \cdot \boldsymbol{x} &= 0 \\ &\cdots\cdots \\ \boldsymbol{l}^{m-\nu} \cdot \boldsymbol{x} &= 0. \end{aligned}$$

Solving algebraically we should get

$$\boldsymbol{x} = t_1 \boldsymbol{p}^1 + t_2 \boldsymbol{p}^2 + \cdots + t_\nu \boldsymbol{p}^\nu$$

with linearly independent *integral* vectors $\boldsymbol{p}^1, \boldsymbol{p}^2, \cdots, \boldsymbol{p}^\nu$ and $t_1, t_2, \cdots, t_\nu$ running through all numbers; and this contradicts the fact that the lattice of integral points in R has at most the dimension $\nu - 1$. The argument therefore shows that the lattice M' has at most the dimension $m - \nu - 1$.

To complete the proof we use RIESZ's theorem (see 5) according to which $\overline{M} = (M')'$. Since M' is a lattice of dimension at most $m - \nu - 1$, the largest vector-subspace in $\overline{M}$ has at least the dimension $m - (m - \nu - 1) = \nu + 1$ (for if M' is of dimension s, its orthogonal space is of dimension $m - s$ and this orthogonal space is obviously contained in the dual module $(M')' = \overline{M}$). This proves our theorem.

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